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Branonium

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Abstract: We study the bound states of brane/antibrane systems by examining the motion of a probe antibrane moving in the background fields of N source branes. The classical system resembles the point-particle central force problem, and the orbits can be solved by quadrature. Generically the antibrane has orbits which are not closed on themselves. An important special case occurs for some Dp-branes moving in three transverse dimensions, in which case the orbits may be obtained in closed form, giving the standard conic sections but with a nonstandard time evolution along the orbit. Somewhat surprisingly, in this case the resulting elliptical orbits are *exact* solutions, and do not simply apply in the limit of asymptotically-large separation or non-relativistic velocities. The orbits eventually decay through the radiation of massless modes into the bulk and onto the branes, and we estimate this decay time. Applications of these orbits to cosmology are discussed in a companion paper.

Keywords: Superstrings and Heterotic Strings, D-branes, Tachyon Condensation.

Contents

1. Introduction

One of the most important developments in string theory of late has been the realization that it contains many kinds of extended objects like Dp-branes. In many situations these objects can be treated much like elementary particles, in that they carry a generalised mass and charge and these give rise to mutual forces which govern their relative motion.

This observation is all the more interesting when the dimension of the brane exceeds three, since then the moving branes can contain an entire world inside of themselves (possibly including us). In this kind of picture the collision of Dp-branes can be interpreted as colliding universes. This observation has recently been used to propose several novel scenarios for the very-early universe, giving rise to inflation or alternatives to inflation [1]–[9], allowing in both cases an attractive geometrical and stringy interpretation of the various cosmologically-active scalar fields.

Perhaps the best-studied brane interactions are brane collisions, which in the particular case of branes and antibranes can lead to brane annihilation (for which insights have come from recent studies of tachyon condensation). However these kinds of direct collisions only begin to explore the rich kinds of physics which are possible when particles and branes mutually interact. A different — and for Dp-branes as yet largely unexplored — possibility is the development of bound orbits, such as commonly arise for particles interacting through central forces.

In this paper we begin the study of the properties of the brane generalisation of such bound orbits. We do so partly motivated by the potential applications of such a study to brane-world cosmology. In particular we could imagine the brane which makes up our own universe being in a closed orbit around another brane due to the mutually attractive forces which act between them. But regardless of these applications, we regard brane bound states to be of interest in their own right inasmuch as they add to our understanding of how branes and strings can interact.

We start this study here with a discussion of the bound states which are possible for brane/antibrane systems. These are among the simplest to use in applications, partly because of the triviality of the various consistency conditions which would otherwise arise when trying to construct brane configurations inside compact extra dimensions. We call this kind of bound state 'branonium' due to its obvious similarity with the analogous electron/positron system: positronium.

Our discussion is couched in terms of a test antibrane which is moving within the fields sourced by a system of N parallel branes. We set up the effective brane lagrangian for this motion and analyse the features of the resulting bound orbits. It happens that despite the many complications of the resulting effective lagrangian, it is still possible to solve explicitly for the trajectory of the extended test brane, with a result which is very similar to those obtained for nonrelativistic motion in a central force. In particular we find the conditions for the existence of stable closed orbits, and show that these include nontrivial systems of D-branes, with perfectly elliptical (non-precessing) orbits being possible if the branes move in three transverse dimensions.

Finally, we give a preliminary study of the stability of the system due to different sources, including radiation reaction due to radiation of bulk and brane modes, as well as nonperturbative instabilities and tachyonic modes. The dominant instability appears to be towards radiation into massless modes, and we compute the orbital decay time for this process.

2. Probe-brane analysis

We start by setting up and solving the equations which are relevant to the motion of a probe antibrane moving within the background of N parallel branes. To do so we first examine the fields which the N source branes set up, and then examine the equations of

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motion which they imply for a test brane or antibrane. We show how these equations may be solved for the motion of the test-brane centre of mass when all other brane modes are frozen. (We return to the question of *why* these modes should be frozen in the next section.) Finally we examine the bound-state orbits and identify the surprising special class for which the exact *relativistic* orbits are ordinary conic sections, and we study the properties of the orbits in this special case in some detail.

2.1 The fields due to N source branes

Let us first consider the gravitational, dilaton and 'electromagnetic' fields which are set up by a set of N parallel p-branes in D-dimensional spacetime. We take the action for these fields to be given in the Einstein Frame by

$$S_s = -\int d^D x\, \sqrt{-g}\left[\frac{1}{2}\, g^{MN}\Big(R_{MN} + \partial_M \phi\, \partial_N \phi\Big) + \frac{1}{2\,n!}\, e^{\alpha\phi}\, F_{M_1 \ldots M_n}\, F^{M_1 \ldots M_n}\right], \qquad (2.1)$$

where the n-form field strength is related to its $(n-1)$-form gauge potential in the usual way $F_{[n]} = dA_{[n-1]}$. Here n is related to the spacetime dimension d of the N p-branes by $d = p + 1 = n - 1$. We denote the d coordinates parallel to the branes by x^μ and the $D - d$ transverse coordinates by y^m.

The constant α depends on which kind of brane is being considered. For instance, if $A_{[n-1]}$ were to arise from a string-frame action

$$S_{\rm SF} = -\frac{1}{2\,n!}\int d^D x\, \sqrt{-\hat g}\; e^{\alpha_s \phi}\, F_{\hat M_1 \ldots \hat M_n}\, F^{\hat M_1 \ldots \hat M_n}\,, \qquad (2.2)$$

with the string-frame metric given by $\hat g_{MN} = e^{\lambda\phi}\, g_{MN}$ for $\lambda = 4/(D-2)$, then $\alpha = \alpha_s + 2 - \lambda(n-1)$. Two cases of particular interest are: (i) NS-NS fields (like the fields rising in the gravity supermultiplet in various dimensions), for which $\alpha_s = -2$ and so $\alpha = \alpha_{NS} = -4(n-1)/(D-2)$; (ii) R-R fields, for which $\alpha_s = 0$ and so $\alpha = \alpha_R = 2(D-2n)/(D-2)$. In the special case $D = 10$ we have $\lambda = \frac{1}{2}$ and so these two cases become $\alpha_{NS} = -\frac{1}{2}(n-1) = -\frac{1}{2}(p+1)$ or $\alpha_R = \frac{1}{2}(5-n) = \frac{1}{2}(3-p)$, respectively.

The solutions to the equations for this system which describe the fields due to a stack of N parallel collocated p-branes is (in the Einstein Frame) [10]:

$$ds^2 = h^{-\tilde\gamma} dx^2 + h^{\gamma} dy^2\,, \qquad e^{\phi} = h^{\kappa}\,, \qquad A_{01\ldots p} = \zeta(1 - h^{-1})\,, \qquad (2.3)$$

where all the other components of the $(n-1)$-form field vanish. The constants γ, $\tilde\gamma$, κ and ξ are given by

$$\gamma = \frac{4d}{(D-2)\,\Delta}\,, \qquad \tilde\gamma = \frac{4\tilde d}{(D-2)\,\Delta}\,, \qquad \kappa = \frac{2\alpha}{\Delta}\,, \qquad \zeta = \frac{2}{\sqrt{\Delta}}\,, \qquad (2.4)$$

where $\tilde d = D - d - 2$ and

$$\Delta = \alpha^2 + \frac{2\,d\,\tilde d}{(D-2)}\,. \qquad (2.5)$$

We assume throughout the existence of more than 2 transverse dimensions, so $\tilde{d} > 0$. The quantity h denotes the harmonic function:

$$h(r) = 1 + \frac{k}{r^{\tilde{d}}}, \qquad (2.6)$$

where r is the radial coordinate transverse to the branes, $r^2 = \delta_{mn}\, y^m y^n$, and k is an integration constant.

Dp-branes in 10 dimensions represent a particularly interesting special case of this solution, corresponding [11] to the choices $\alpha = \alpha_R = \frac{1}{2}(3-p)$, $\Delta = 4$, $d = p+1$, $\tilde{d} = 7-p$, $\kappa = \frac{1}{4}(3-p)$ and $k = c_p\, g_s N\, l_s{}^{\tilde{d}}$. (Clearly the condition $\tilde{d} > 0$ then implies $p < 7$.) Here l_s is the string length, which is related to the string tension by $l_s{}^2 = \alpha'$. The string coupling constant at infinity is denoted g_s and c_p is the constant $c_p = (2\sqrt{\pi})^{5-p}\,\Gamma(\frac{7-p}{2})$.

Having an explicit string solution is useful for many reasons, not the least of which is the understanding which this gives of the limits of validity of the field theoretic description we shall use throughout what follows. For these purposes there are two dimensionless quantities which are relevant. The first of these controls the validity of string perturbation theory, which for string self-interactions requires the local string coupling, $g(r) = g_s\, e^{\phi}$, to be everywhere small. The smallness of this quantity is required in order to suppress the modifications which higher string loops would otherwise have on our semiclassical analysis.

The second important dimensionless quantity is the ratio r/l_s, which must be large in order to believe that the brane interactions are dominated only by the mediation of bulk fields which correspond to massless string states. Once $r \sim l_s$ higher order corrections in α' become important because over these short distances contributions from higher string modes become possible. A particularly dangerous example of such a correction arises if a probe antibrane is introduced, since then there is an open string tachyon which appears at short distances within the open-string sector for strings which stretch between the N branes and the antibrane. Within the closed string picture this tachyon can be regarded as a kind of α' correction, since by duality it can be thought of as emerging when all the closed string modes are resummed. The appearance of this tachyon signals the onset of the instability in which the probe antibrane annihilates with one of the source branes.

One might worry that the requirement $r \gg l_s$ might also preclude using the full nonlinear solution given above, requiring us instead to use only its leading approximation in powers of $k/r^{\tilde{d}}$. But this need not be required if the solution's characteristic length scale, $\rho = k^{1/\tilde{d}} = l_s(c_p\, g_s N)^{1/(7-p)}$, is also large, $\rho \gg l_s$, since in this case we may have $k/r^{\tilde{d}} = O(1)$ even if $r \gg l_s$. Notice that for Dp-branes the condition $\rho \gg l_s$ requires $g_s N \gg 1$, and so takes us beyond the limit of open-string perturbation theory, which requires $g_s N \ll 1$ (where the factor of N here arises from the Chan-Paton contributions from each world sheet boundary). In this way the field-theory limit provides (for $r \gg l_s$) results which are nonperturbative in the coupling of the bulk fields to the branes.

2.2 The probe brane

In the above background let us now follow the motion of another single parallel p-brane, displaced from the original N by the radial coordinate-distance r (for a related discussion see [14]). This motion is described by the brane action, which can be decomposed as the sum of two pieces: the Born-Infeld (BI) and the Wess-Zumino (WZ) parts. The dilaton and graviton couplings are given by the Born-Infeld contribution, which is (in the String Frame):

$$S_{\rm BI} = -T_p \int d^d\xi \; e^{-\phi} \sqrt{-\det(\hat{g}_{MN}\, \partial_\mu x^M \partial_\nu x^N + 2\, {l_s}^2 \mathcal{F}_{\mu\nu})}\,, \tag{2.7}$$

where x^M are the coordinates of the embedding, ξ^μ are the world-volume coordinates and T_p denotes the brane tension. $\mathcal{F}_{\mu\nu}$ denotes the field strength for any open-string gauge modes confined to the brane, although for the moment we put these fields to zero.

The coupling to the bulk gauge field, $A_{[p+1]}$, is given by the Wess-Zumino part of the brane action:

$$S_{WZ} = -q\, T_p \int A_{[p+1]}\,, \tag{2.8}$$

where q represents the brane charge, with $q = 1$ representing a probe brane and $q = -1$ representing a probe antibrane.

Since our branes are straight and parallel it is convenient to choose the following coordinate gauge:

$$x^\mu = \xi^\mu\,, \tag{2.9}$$

where as before x^μ are the coordinates parallel to the world volume. To the extent that we follow only the brane's overall centre-of-mass motion — we return to the validity of this assumption in the next section — we may also take

$$y^m = y^m(t)\,, \tag{2.10}$$

where $t = x^0$ denotes coordinate time and $y^m(t)$ are the coordinates perpendicular to the brane surface, which parameterise the position of the probe. With these choices the induced String-Frame metric on the probe becomes:

$$d\hat{s}^2 = e^{\lambda\phi}\, ds^2 = h^{-\beta} \left[\left(-1 + h^\omega v^2\right) dt^2 + (d\xi^i)^2 \right]\,, \tag{2.11}$$

where $i = 1, \dots, p$ and $v^2 = \sum_m (dy^m/dt)^2$. The constants β and ω are given by

$$\beta = \tilde{\gamma} - \kappa\lambda = \frac{4(\tilde{d} - 2\alpha)}{(D-2)\,\Delta}\,, \qquad \omega = \gamma + \tilde{\gamma} = \frac{4}{\Delta}\,. \tag{2.12}$$

For the special case of Dp-branes in 10 dimensions these become $\beta = 1/2$ and $\omega = 1$.

The effective action for the probe brane is then:

$$S = -T_p\, V_p \int dt \; \left[\frac{1}{h^\eta} \sqrt{1 - h^\omega\, v^2} + q\, \zeta \left(1 - \frac{1}{h}\right) \right]\,, \tag{2.13}$$

where $V_p = \int d^p\xi$ is the spatial volume spanned by the probe brane, $\zeta = \sqrt{\omega}$ given by eq. (2.4) and

$$\eta = \kappa + \frac{\beta\, d}{2} = \frac{2\,[\alpha(\tilde{d} - d) + d\,\tilde{d}]}{(D-2)\,\Delta}\,. \tag{2.14}$$

For the special case of Dp-branes in 10 dimensions we therefore have $\eta = 1$.

Apart from an additive constant, the brane action has the form

$$S = -m \int dt \,\left[\frac{1}{h^\eta}\sqrt{1 - h^\omega\, v^2} - \frac{\hat{q}}{h}\right] \tag{2.15}$$

with $\hat{q} = \zeta\, q$ and $m = T_p V_p$. This is very similar to that of a particle moving in the $D - d$ dimensions transverse to the branes. That part of the action containing the square root corresponds to the particle's relativistic kinetic energy and the coupling to the dilaton field, while the part involving $\hat{q}$ corresponds to the potential due to the Ramond-Ramond couplings. Notice that the apparent mass, m, of the particle scales with the brane world-volume, and so becomes infinitely large in the limit that this volume diverges.

Several features of the probe-brane action bear emphasis:

- A measure of the potential energy of the probe brane may be obtained by evaluating the brane action for a brane at rest: $v = 0$. The result obtained in this way is proportional to the combination $h^{-\eta} - \hat{q}/h$. For Dp-branes in 10 dimensions this becomes $(1-q)/h$, which is nonzero for a probe antibrane ($q = -1$), but vanishes for probe brane ($q = 1$). This is as one expects given the BPS nature of the Dp-brane.

- The brane motion is along a null geodesic in the transverse dimensions when $v^2 = h^{-\omega}$. This becomes $v = 1$ asymptotically far from the set of N source branes, but is smaller than unity closer to the source branes.

- Massless particles trapped on the probe brane move along null geodesics of the induced metric, which corresponds to $(d\xi^i/dt)^2 = 1 - h^\omega v^2$, indicating that the effective maximum speed for trapped particles, $c^2 = 1 - h^\omega v^2$, varies with time as the brane moves through the fields of the source branes. Notice that $c^2 \to 0$ if the brane becomes ultra-relativistic: $v^2 \to h^{-\omega}$.

2.3 Classical dynamics

Let us analyse the movement in $d_T = D - d$ dimensions of a particle with the lagrangian:

$$L = -m\left(\frac{1}{h^\eta}\sqrt{1 - h^\omega\, v^2} - \frac{\hat{q}}{h}\right), \tag{2.16}$$

with $h = 1 + k/r^{\tilde{d}}$ and $k > 0$.

Invariance of this lagrangian under $O(d_T)$ rotations ensures the conservation of the angular momentum tensor, $L_{ij} = x_i\, p_j - x_j\, p_i$, with x_i and p_j denoting the components of the particle's position and canonical momentum vectors. The conservation of this tensor ensures that the particle motion is confined to the plane which is spanned by the particle's initial position and momentum vectors. Let us denote by r and θ polar coordinates in the plane of the particle motion. The lagrangian for the resulting two-dimensional system is

$$L = -m\left(h^{-\eta}\sqrt{1 - h^{\omega}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)} - \hat{q}\,h^{-1}\right). \tag{2.17}$$

Invariance under translations of θ ensures the conservation of angular momentum

$$\ell = \frac{p_\theta}{m} = \frac{r^2\dot{\theta}}{h^{\eta-\omega}\sqrt{1 - h^{\omega}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}}. \tag{2.18}$$

Notice that this is just the relativistic angular momentum. In terms of this the angular velocity becomes

$$r^2\dot{\theta}^2 = \frac{\ell^2\,h^{2(\eta-\omega)}(1 - h^{\omega}\,\dot{r}^2)}{r^2 + \ell^2\,h^{2\eta-\omega}}. \tag{2.19}$$

The canonical momentum associated with the radial motion is similarly

$$\begin{aligned}\rho_r = \frac{p_r}{m} &= \frac{\dot{r}}{h^{\eta-\omega}\sqrt{1 - h^{\omega}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}}, \\ &= \frac{\dot{r}}{h^{\eta-\omega}}\left(\frac{1 + \ell^2\,h^{2\eta-\omega}/r^2}{1 - h^{\omega}\,\dot{r}^2}\right)^{1/2}.\end{aligned} \tag{2.20}$$

Inverting this to obtain $\dot{r}$ as a function of ρ_r then gives

$$\dot{r}^2 = \frac{h^{2(\eta-\omega)}\,\rho_r^2}{1 + h^{2\eta-\omega}\left(\ell^2/r^2 + \rho_r^2\right)}. \tag{2.21}$$

Constructing the canonical hamiltonian gives the energy formula:

$$\varepsilon = \frac{E}{m} = \frac{1}{h^{\eta}}\left[1 + h^{2\eta-\omega}\left(\frac{\ell^2}{r^2} + \rho_r^2\right)\right]^{1/2} - \frac{\hat{q}}{h}, \tag{2.22}$$

which is conserved during the motion.

2.3.1 Qualitative analysis

A qualitative understanding of the orbits which are implied by this system may be obtained using the effective potential

$$V_{\text{eff}}(r) = \varepsilon(\rho_r = 0) = \frac{1}{h^{\eta}}\left[1 + h^{2\eta-\omega}\left(\frac{\ell^2}{r^2}\right)\right]^{1/2} - \frac{\hat{q}}{h}, \tag{2.23}$$

which we plot in figure 1 for the case $p = 6$.

The utility of this potential follows from the observation that ε is a monotonically increasing function of ρ_r: $\partial\varepsilon/\partial\rho_r \geq 0$. The allowed range of r for classical motion is therefore easily found by the standard device of plotting $V_{\text{eff}}(r)$ against r, and finding those r's for which $\varepsilon \geq V_{\text{eff}}(r)$.

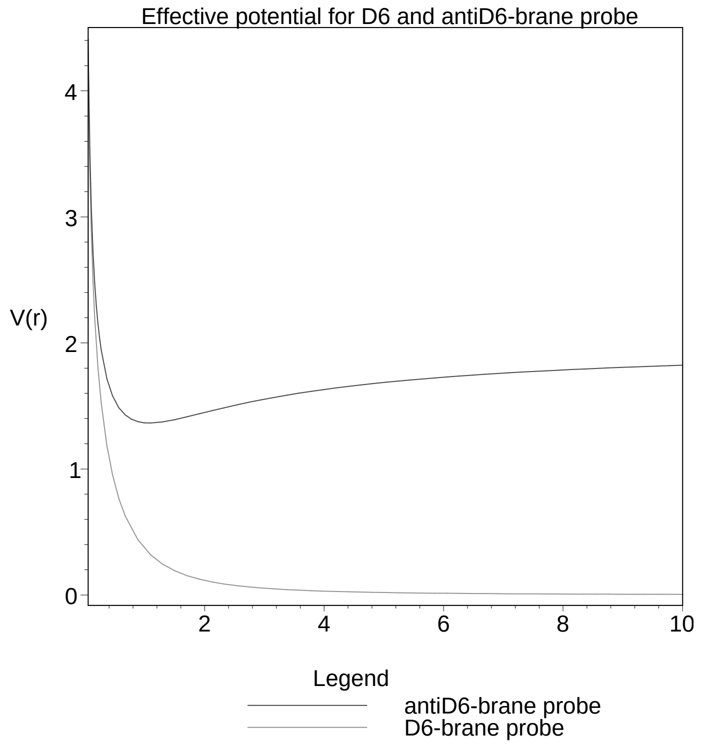

Figure 1: The effective potential for the radial motion of the brane centre-of-mass for two different probes, a D6 and an anti-D6-brane in the background of a large number N of D6-branes. We have taken $Ng_sl_s = 2$. Bound states exist only for anti-D6-branes.

The behaviour of $V_{\rm eff}$ follows from the properties of h, which is a monotonically decreasing function of r, with limits $h \to k/r^{\tilde{d}} \to \infty$ as $r \to 0$ and $h \to 1$ as $r \to \infty$. This implies

$$V_{\rm eff} \to (1-\hat{q}) + \begin{cases} (\hat{q}-\eta)k/r^{\tilde{d}} & \text{for } \tilde{d} < 2 \\ [\ell^2/2 + \hat{q} - \eta]k/r^2 & \text{for } \tilde{d} = 2 \\ \ell^2/(2r^2) & \text{for } \tilde{d} > 2 \end{cases} \qquad \text{as } r \to \infty\,, \tag{2.24}$$

and, provided η and ω are positive,

$$V_{\rm eff} \to \begin{cases} \ell r^{-1+\omega\tilde{d}/2}/k^{\omega/2} & \text{for } (\omega-2)\,\tilde{d} < 2 \\ [\ell/k^{\omega/2} - \hat{q}/k]r^{\tilde{d}} & \text{for } (\omega-2)\,\tilde{d} = 2 \\ -\hat{q}r^{\tilde{d}}/k & \text{for } (\omega-2)\,\tilde{d} > 2 \end{cases} \qquad \text{as } r \to 0\,. \tag{2.25}$$

These limiting forms are often sufficient to determine the existence of a minimum for $V_{\rm eff}$ for some value $r = r_c$, which signals the existence of a stable circular orbit at this radius. For instance, for the 10 dimensional Dp-brane example we have $\omega = \eta = 1$ and $\tilde{d} = 7 - p > 0$. This then gives the limits

$$\begin{aligned} V_{\rm eff} &\to (1-q) + \begin{cases} (q-1)k/r & \text{for } p = 6 \\ [\ell^2/2 + q - 1]k/r^2 & \text{for } p = 5 \\ \ell^2/(2r^2) & \text{for } p < 5 \end{cases} \qquad \text{as } r \to \infty\,, \\ V_{\rm eff} &\to \ell r^{(5-p)/2}/k^{1/2} \qquad \text{as } r \to 0\,. \end{aligned} \tag{2.26}$$

These expressions show that for $p > 5$ $V_{\rm eff}$ blows up (to positive infinity) at the origin, and goes asymptotically to a constant as $r \to \infty$. Furthermore, if $q = -1$ then $V_{\rm eff}$ begins decreasing as r comes in from infinity, and so there must be a minimum somewhere in between. The position, r_c, of the minimum is the solution to the equation $(2r_c+1)^2 \ell^2 = 8r_c^3$.

For $p < 5$, $V_{\rm eff}$ instead approaches zero both as $r \to 0$ and as $r \to \infty$, and it grows as r comes in from infinity. The effective potential in this case instead has a maximum at some intermediate radius, and no stable circular orbits exist. Bound motion exists for distances smaller than the one corresponding to the maximum of the potentials but the trajectories are not closed and eventually pass through the origin.

The angular frequency, Ω, for a circular orbit is a constant, which is related to the orbital radius, r_c, by eq. (2.19) specialized to $\dot{r} = 0$:

$$\Omega^2 = \frac{\ell^2 \, h_c^{2(\eta-\omega)}}{r_c^2 \left(r_c^2 + \ell^2 \, h_c^{2\eta-\omega} \right)}, \tag{2.27}$$

with $h_c = h(r_c)$.

These circular orbits, when they exist, have the minimum energy for a given angular momentum, and so are stable (at least within the present framework). If the energy is increased slightly, for fixed ℓ, a broader class of orbits arises for which $\rho_r \neq 0$ and so for which the radial coordinate oscillates between the nearest classical turning points, $r_\pm$, defined as the two roots (on either side of r_c) of the condition $\varepsilon = V_{\rm eff}(r_\pm)$. For $p > 5$ the resulting orbits remain bounded away from $r = 0$ since they oscillate around the radius of the stable circular orbit. For $p < 5$ by contrast the bound orbits have radii which range from zero up to a turning point closer to $r = 0$ than the position of the maximum of $V_{\rm eff}$.

In general these orbits are not closed, since the period of the radial oscillations need not be commensurate with the angular period. In this case the orbits will be ergodic in the sense that the system eventually passes through all points which are allowed by the given conserved energy and angular momentum. Since for $p < 5$ these orbits include radii near $r = 0$, the probe brane can and will pass near the source branes, and so is likely to be quickly annihilated.

The existence of such unclosed orbits is not a surprise, given that similar orbits also arise for solutions to the relativistic Kepler problem (such as for the precession of the orbit of Mercury). What we shall find which *is* a surprise is that for some brane systems the orbits *are* closed non-precessing ellipses, even once relativistic effects are included.

2.3.2 Explicit solutions

We now turn to a more explicit characterization of the orbits. Notice that the above equations of motion imply that the derivative $dr/d\theta$ takes the simple form:

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{r^2 \, \rho_r}{\ell} \,. \tag{2.28}$$

To solve for the trajectory followed by the particle we start from eq. (2.28) and use eq. (2.22) to eliminate ρ_r in favour of the conserved ε. Using the standard change of variable, $u = 1/r$, we have $u' = du/d\theta = -\rho_r/\ell$ and so

$$\varepsilon = \frac{1}{h^\eta}\left(\left[1 + h^{2\eta-\omega}\ell^2\left(u^2 + u'^2\right)\right]^{1/2} - \frac{\hat{q}}{h}\right), \tag{2.29}$$

where now $h(u) = 1 + k\,u^{\tilde{d}}$. The orbit is then obtained by quadrature:

$$\theta - \theta_0 = \ell \int_{u_0}^{u} \frac{dx}{F(x)}, \tag{2.30}$$

where

$$F(x) = \left[\varepsilon^2\, h^\omega(x) + 2\varepsilon\,\hat{q}\, h^{\omega-1}(x) + \hat{q}^2\, h^{\omega-2}(x) - h^{\omega-2\eta}(x) - \ell^2 x^2\right]^{1/2}. \tag{2.31}$$

2.3.3 Dp-branes

Again 10 dimensional Dp-branes provide a particularly interesting special case, corresponding to the choices $\eta = \omega = 1$ and $\hat{q} = q = \pm 1$. In this case the lagrangian (2.16) becomes

$$L = -\frac{m}{h}\left[\sqrt{1 - h\,(\dot{r}^2 + r^2\dot{\theta}^2)} - q\right], \tag{2.32}$$

and so the canonical momenta become

$$\ell = \frac{r^2\dot{\theta}}{\sqrt{1 - h\,(\dot{r}^2 + r^2\dot{\theta}^2)}}, \tag{2.33}$$

and

$$\rho_r = \dot{r}\left(\frac{1 + \ell^2\,h/r^2}{1 - h\,\dot{r}^2}\right)^{1/2}. \tag{2.34}$$

The conserved energy then is

$$\varepsilon = \frac{1}{h}\left\{\left[1 + h\left(\frac{\ell^2}{r^2} + \rho_r^2\right)\right]^{1/2} - q\right\}. \tag{2.35}$$

In this case eqs. (2.30) and (2.31) defining the orbits become

$$\theta - \theta_0 = \int_{u_0}^{u} \frac{dx}{\sqrt{A + B\,x^{\tilde{d}} - x^2}}, \tag{2.36}$$

where $A = (\varepsilon^2 + 2\,q\,\varepsilon)/\ell^2$ and $B = k\,\varepsilon^2/\ell^2$. What is striking about this solution is that it is precisely the same orbit as would be given by classical nonrelativistic motion in the presence of a central potential of the form $V_c = -m_c k_c/r^{\tilde{d}}$:

$$\frac{L_c}{m_c} = \frac{1}{2}\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) + \frac{k_c}{r^{\tilde{d}}}. \tag{2.37}$$

The orbits for this lagrangian [12] are given by eq. (2.36), but with $A = A_c = 2\varepsilon_c/\ell_c^2$ and $B = B_c = 2k_c/\ell_c^2$, where as before $\varepsilon_c = E_c/m_c$ and $\ell_c = p_\theta/m_c$ are respectively the energy and angular momentum per unit mass. These correspond to replacing $k_c = 2\,k$ and taking the nonrelativistic limit: $\varepsilon = 2 + \varepsilon_c$ with $|\varepsilon_c| \ll 1$.

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This shows that the orbits of the fully relativistic Dp-brane system have *exactly* the same shape as those of the nonrelativistic system — given by eq. (2.37) — with the corresponding orbital parameters read off by making the substitution $(A_c, B_c) \leftrightarrow (A, B)$. In particular, for bound states we take $\varepsilon_c < 0$ and so have $A_c < 0$. It follows that the bound Dp-brane orbits must also have $A < 0$. Now for Dp-branes the energy expression, eq. (2.35), implies $\varepsilon \geq 0$ for both $q = \pm 1$. Consequently $A = (\varepsilon^2 + 2\,q\,\varepsilon)/\ell^2$, can only be negative for antibranes, since then $q = -1$. This is as expected since only if the probe is an antibrane does it experience an attractive net force.

It is quite remarkable that the orbits for this system are exactly solvable at the same level as are those of the analogous nonrelativistic central force systems, particularly given that the Dp-brane lagrangian includes an infinite number of terms when written in powers of $1/r$ and v^2. We note in passing that although the shape of the orbits are unchanged from the nonrelativistic central-force problem, the time evolution of the branes along the orbits *does* change, as will be seen in more detail in a later section.

2.3.4 Branes at large-distances

To this point we have studied the orbits using the full solutions to the nonlinear field equations, which for Dp-branes can be done within a controlled approximation provided $g_s N \gg 1$. We now turn to the case where r is sufficiently large that only the weak-field form for the fields is required. This might be expected to apply in the case where N is not large.

For large r the virial theorem states that $\frac{1}{2}\,v^2 \sim k/r^{\tilde{d}}$ and so if we expand in powers of $k/r^{\tilde{d}}$ we must also expand to the same order in powers of v^2. Expanding the lagrangian, eq. (2.16), in this way gives (up to an additive constant)

$$S = T_p V_p \int dt \Bigg\{ \left[(\eta - \hat{q})\,\frac{k}{r^{\tilde{d}}} + \frac{v^2}{2} \right] + \\ + \left[\left(\hat{q} - \frac{\eta\,(\eta+1)}{2} \right) \left(\frac{k}{r^{\tilde{d}}} \right)^2 + (\omega - \eta)\,\frac{k\,v^2}{r^{\tilde{d}}} + \frac{v^4}{8} \right] + \cdots \Bigg\}. \tag{2.38}$$

For Dp-branes this reduces to

$$S = T_p V_p \int dt \left\{ \left[(1-q)\,\frac{k}{r^{\tilde{d}}} + \frac{v^2}{2} \right] + \left[(q-1) \left(\frac{k}{r^{\tilde{d}}} \right)^2 + \frac{v^4}{8} \right] + \cdots \right\}, \tag{2.39}$$

so we see that the leading dynamics is that of a particle moving in the presence of the central potential $V = (q-1)k/r^{\tilde{d}}$, which is attractive for $q = -1$. This is what is expected from explicit calculations using the string cylinder graph. The orbits for this case are given by eq. (2.36), but with $A = A_c$ and $B = B_c$.

We find in this way that eq. (2.36) describes Dp-brane orbits for all $r \gg l_s$. For $g_s N \gg 1$ the parameters A and B are as given below eq. (2.36), and for $g_s N \lesssim 1$ this becomes $A \approx A_c$ and $B \approx B_c$. Although one might have expected the inclusion of the higher-order corrections in powers of v^2 and $k/r^{\tilde{d}}$ to more dramatically change the form of the orbits, this does not happen. Because the conditions $\eta = \omega = \zeta = 1$ which ensure this simplicity for the brane/antibrane orbits ($q = -1$) also ensures the complete absence of an interaction potential between branes ($q = +1$), we conjecture that the independence of the orbit shapes from receiving relativistic corrections arises only for BPS branes and antibranes. A deeper understanding of when this occurs, and whether there is a connection with the BPS nature of the Dp-branes involved, would clearly be worthwhile.

2.3.5 Closed orbits

We next demonstrate the existence of brane orbits which close on themselves. Our interest here is not the circular closed orbits, which close on themselves in a trivial way and which arise for many kinds of branes. Rather we look here for the less trivial situation where the orbits vary in both the angular and radial directions, but these motions have commensurate periods.

The existence of such orbits is suggested by the following two observations. First, we have seen that some relativistic branes (like the BPS Dp-branes in 10 dimensions) have precisely the same orbits as has the nonrelativistic central-force problem. Second, we know that motion in the nonrelativistic Coulomb potential, $V \propto 1/r$, is along conic sections, with the bound orbits being ellipses. We focus therefore on the case of source D6-branes and a probe D6-antibrane, for which $\tilde{d} = 7 - p = 1$ and $q = -1$.

The probe-brane orbits are described by eq. (2.36), which for $\tilde{d} = 1$ may be integrated in closed form in terms of elementary functions. When $A < 0$ the result is an ellipse

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}\,, \tag{2.40}$$

where the orbit's semimajor axis, a, and eccentricity, e, are given by

$$\begin{aligned} a &= -\frac{B}{2A} = \frac{k\,\varepsilon}{2\,(2-\varepsilon)}, \\ e &= \left[1 + \frac{4A}{B^2}\right]^{1/2} = \left[1 - \frac{4\ell^2\,(2-\varepsilon)}{k^2\varepsilon^3}\right]^{1/2}. \end{aligned} \tag{2.41}$$

For comparison, using the nonrelativistic values $A_c = 2\varepsilon_c/\ell_c^2$ and $B_c = 2k_c/\ell_c^2$ instead gives the usual expressions $a_c = -k_c/(2\varepsilon_c)$ and $e_c = [1 + 2\,\varepsilon_c\ell_c^2/k_c^2]^{1/2}$.

The time evolution along this orbit is obtained by integrating equation (2.14), giving

$$t - t_0 = \int_{r_0}^{r} dx\, \frac{\Big[(\varepsilon + q)\,x + \varepsilon\,k\Big]}{\sqrt{(\varepsilon + 2\,q)\,\varepsilon x^2 + \varepsilon^2 k\,x - \ell^2}} \tag{2.42}$$

which can be integrated to give

$$\Omega t = \psi - \tilde{e}\sin\psi \tag{2.43}$$

with the angle ψ denoting the *eccentric anomaly*, defined by $r = a(1 - e\cos\psi)$ or, equivalently,

$$\cos\psi = \frac{a-r}{ae} = \frac{k\varepsilon^2 - 2\,\varepsilon(2-\varepsilon)\,r}{\sqrt{k^2\,\varepsilon^4 - 4\,\ell^2\,\varepsilon(2-\varepsilon)}}\,. \tag{2.44}$$

The angles ψ and θ are explicitly related by $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2}$. Finally, the constant $\tilde{e}$ which appears in (2.43) is:

$$\tilde{e} = \left(\frac{\varepsilon - 1}{3-\varepsilon}\right) e = \frac{\varepsilon - 1}{3-\varepsilon}\sqrt{1 - \frac{4\ell^2(2-\varepsilon)}{k^2\varepsilon^3}}\,. \tag{2.45}$$

The period of the orbit is given by:

$$T = \frac{2\pi}{\Omega} = \pi k\left(\frac{3-\varepsilon}{2-\varepsilon}\right)\sqrt{\frac{\varepsilon}{2-\varepsilon}} \;=\; \pi\,\sqrt{\frac{a}{2k}}\,(2a+3k)\,. \tag{2.46}$$

It is instructive to compare this with the nonrelativistic Kepler problem. In this case the time dependence is given by a relation similar to eq. (2.43):

$$\Omega_c t = \psi - e_c\,\sin(\psi)\,, \tag{2.47}$$

where ψ is related to θ as before. The orbital period which results is $T_c = 2\pi/\Omega_c = 2\pi\sqrt{a_c^3/k_c}$.

We see that the D6-brane and the nonrelativistic problems share the same functional form for both $t(\psi)$ and $r(\theta)$, but the parameters which appear in the two solutions differ. (For instance $\tilde{e} \neq e$ for the D6-brane, while $\tilde{e}_c = e_c$ nonrelativistically.) This difference in parameters for the two solutions implies a different relationship is obtained between orbital features like a and e and the underlying physical quantities, like ε and ℓ and m.

Although the orbits have the same shape, the differences in their parametric dependence on ε and ℓ also implies differences for how the period relates to the orbital shape and size. For instance neither Kepler's second nor third laws apply for D6-branes, since T^2/a^3 is not a universal quantity for all probe brane orbits, and since conservation of ℓ does not imply that the product $r^2\,\dot{\theta}$ is a constant along any one orbit.

2.3.6 The Runge-Lenz vector

Here we explore further the remarkable property that the closed orbits for the nontrivial D6-brane system, described by the lagrangian, (2.32), are identical to the orbits for the standard nonrelativistic $1/r$ central force problem.

For the nonrelativistic Kepler problem, the fact that the orbits are closed ellipses which do not precess is reflected by the existence of an extra conserved quantity: the Laplace-Runge-Lenz vector. Normally the existence of this vector is considered to be a special consequence of the $1/r$ potential, which is not shared by other central-force potentials.

Nonetheless, despite the fact that the D6-brane lagrangian contains many relativistic corrections as well as a potential which is not purely proportional to $1/r$, its orbits are also non-precessing ellipses. One might therefore wonder whether in this case there is also a generalised Runge-Lenz vector whose conservation ensures the absence of precession. We show in this section, by explicit construction, that the answer to this question is 'yes'.

Since there are 3 spatial dimensions transverse to the D6 brane in 10 dimensions, it is convenient to phrase the analysis using the angular momentum vector, which is dual to the conserved antisymmetric tensor. Let us therefore consider the vector:

$$\mathbf{\Lambda} \;\equiv \frac{d}{dt}\left(f(\varepsilon)\;\mathbf{p}\wedge\mathbf{L}\right) = f(\varepsilon)\;\dot{\mathbf{p}}\wedge\mathbf{L}\,, \tag{2.48}$$

where $\mathbf{L} = \mathbf{x} \wedge \mathbf{p}$ is the conserved angular momentum vector and for the moment $f(\varepsilon)$ is an unspecified function of energy to be fixed below. The linear momentum is defined as the canonical quantity

$$\mathbf{p} = \frac{1}{m}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{1}{\sqrt{1 - h\,v^2}}\;\dot{\mathbf{x}}\,, \tag{2.49}$$

and

$$\begin{aligned} \dot{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{x}} &= \frac{1}{h^2\sqrt{1-h\,v^2}}\left[1 - \frac{h\,v^2}{2} - q\sqrt{1-h\,v^2}\right]\;\frac{\partial h}{\partial r}\,\frac{\mathbf{x}}{r} \\ &= \frac{\varepsilon^2\sqrt{1-h\,v^2}}{2m}\,\frac{\partial h}{\partial r}\,\frac{\mathbf{x}}{r}\,, \end{aligned} \tag{2.50}$$

where in the last equality we used the expression for the energy eq. (2.22) as well as $q^2 = 1$. Using this expression to evaluate the quantity $\dot{\mathbf{p}}$ in eq. (2.48) we obtain:

$$\mathbf{\Lambda} = 2f(\varepsilon)\,\varepsilon^2\,\frac{1}{r}\frac{\partial h}{\partial r}\;\left[\mathbf{x} \wedge (\mathbf{x} \wedge \dot{\mathbf{x}})\right]. \tag{2.51}$$

Now if we choose $f(\varepsilon) = 1/\varepsilon^2$ and use the identity

$$\mathbf{x} \wedge (\mathbf{x} \wedge \dot{\mathbf{x}}) \;=\; r\,\dot{r}\,\mathbf{x} - r^2\,\dot{\mathbf{x}} \;=\; -r^3\,\frac{d}{dt}\left(\frac{\mathbf{x}}{r}\right), \tag{2.52}$$

we find

$$\mathbf{\Lambda} = \frac{1}{2}\frac{\partial h}{\partial r}\,r^2\,\frac{d}{dt}\left(\frac{\mathbf{x}}{r}\right). \tag{2.53}$$

Now comes the main point. Only for the particular case $h = 1 + k/r$ is the right-hand-side of this last equation a total derivative, and so can be used to make a conserved quantity — the generalised Runge-Lenz vector:

$$\mathbf{A} = \frac{1}{\varepsilon^2}\;\mathbf{p} \wedge \mathbf{L} \;-\; \frac{k}{2}\,\frac{\mathbf{x}}{r}\,. \tag{2.54}$$

Notice that, as expected, $\mathbf{A} \cdot \mathbf{L} = 0$, and

$$\mathbf{A} \cdot \mathbf{x} = A\,r\cos\theta = \frac{\ell^2}{\varepsilon^2} - \frac{k\,r}{2} \tag{2.55}$$

which exactly reproduces the elliptic trajectory $r(\theta)$ as before, and sets $A = ke/2$.

As in the nonrelativistic case, the Runge-Lenz vector indicates a further symmetry beyond rotation invariance, and because it lives within the plane of the ellipse, pointing along the semimajor axis, its conservation forbids the orbits from precessing.

2.4 Quantum dynamics

We next describe the quantum behaviour which follows from the lagrangian, (2.32). We do so for its intrinsic interest, even though all of our later brane applications will be well within the classical domain.

Recall that for Dp-branes the hamiltonian is the following function of the canonical momentum

$$\mathcal{H} = \frac{1}{h}\;\left(\sqrt{m^2 + h\,p^2} - mq\right). \tag{2.56}$$

The Schrödinger equation for this hamiltonian, $\mathcal{H}\Psi = E\Psi$, therefore becomes

$$\frac{1}{h}\,\left(\sqrt{m^2 + h\,p^2} - m\,q\right)\Psi = E\Psi\,. \tag{2.57}$$

This may be written in a more useful form by isolating and squaring the square root on the left-hand-side, leading to

$$\left(m^2 + h\,p^2\right)\,\Psi \;=\; (Eh + mq)^2\,\Psi\,. \tag{2.58}$$

Because we have squared $\mathcal{H}$ in what follows we will find solutions having both signs for E. Since $\mathcal{H}$ is manifestly positive for $q = \pm 1$, we must drop all solutions for which $E < 0$ (notice that the time-dependent Schrödinger equation will include two time derivatives indicating positive and negative frequencies).

For $q^2 = 1$ and $h = 1 + k/r^{\tilde{d}}$ this can be further manipulated to be written as:

$$\left(p^2 - \frac{E^2\,k}{r^{\tilde{d}}}\right)\,\Psi = E(2mq + E)\Psi\,. \tag{2.59}$$

The utility of this last form comes from its similarity with the corresponding nonrelativistic central-force problem:

$$\left(\frac{p^2}{2\mu} - \frac{g^2}{r^{\tilde{d}}}\right)\,\Psi \;=\; E_c\Psi\,, \tag{2.60}$$

which becomes identical to eq. (2.59) after the replacements $\mu \to 1/2, g^2 \to E^2 k$, and $E_c \to E(E + 2qm)$.

In particular, for D6 branes we have $\tilde{d} = 1$, and so eq. (2.59) has the same form as the Schrödinger equation for the Hydrogen atom, for which the solutions have been known since antiquity. In this case the energy eigenvalues consist of a continuum, $E_c \geq 0$, plus a discrete spectrum of bound states labelled by the quantum number $n = 1, 2, \ldots$: $E_c = -g^4\,\mu/(2n^2)$.[1]

Making the substitutions which convert this spectrum to the D6-brane problem, we find the following energy spectrum:

$$\begin{aligned} E(E + 2qm) &\geq 0\,, \qquad \text{(continuum)} \\ E(E + 2qm) &= -\frac{E^4 k^2}{4n^2}\,, \qquad \text{(discrete)}. \end{aligned}$$

Requiring an energy spectrum which is bounded from below implies the continuum branch becomes $E \geq 0$ if $q = 1$ and $E \geq 2m$ if $q = -1$.

The energies for the discrete spectrum can be explicitly solved to give the following expression in terms of the level n and the constant k:

$$E(n) = \left(\frac{8n^2}{k^2}\right)^{1/3}\left[\left(\sqrt{\frac{m^2}{4} + \frac{n^2}{27k^2}} - \frac{qm}{2}\right)^{1/3} - \left(\sqrt{\frac{m^2}{4} + \frac{n^2}{27k^2}} + \frac{qm}{2}\right)^{1/3}\right]. \tag{2.61}$$

[1] Recall our use of fundamental units, for which $\hbar = c = 1$.

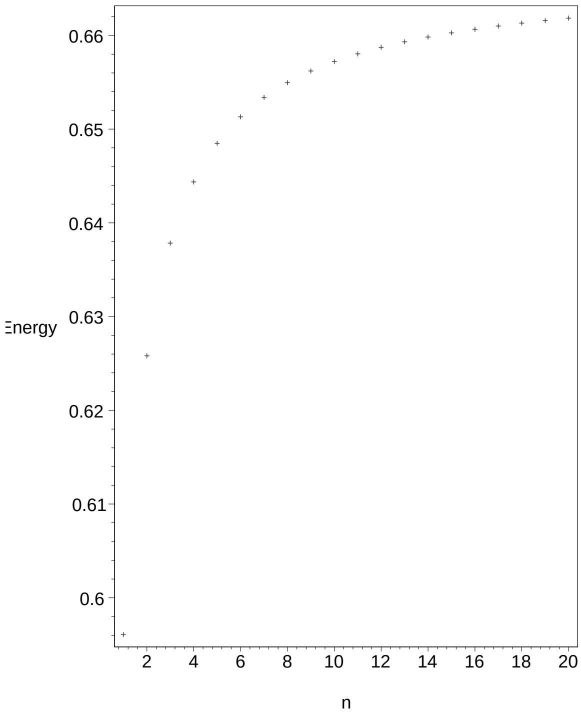

Figure 2: The energy of the first 20 bound states of the branonium. We have taken $m = 1/3$ and $k^2 = 4/3$.

Notice that the square bracket in this expression is negative if $q = +1$ and is positive if $q = -1$. Consequently for $q = +1$ these discrete eigenvalues are spurious and must be discarded. For $q = -1$ the discrete eigenvalues are positive, and approach the continuum ($E = 2m$) as $n \to \infty$. The first 20 energy levels which follow from eq. (2.61) are plotted against n in figure 2.

The characteristic size of the corresponding discrete energy eigenstates in the transverse dimensions is set by the D6-brane analog of the Bohr radius: $a_B = 1/(g^2\,\mu)$. After the appropriate substitutions this becomes

$$a_B = \frac{2}{E^2\,k}\,. \tag{2.62}$$

For subsequent applications it is interesting to see how these expressions behave in the decompactification limit, for which the spatial world-volume, V_6, of the brane becomes very large. Because $m = T_6 V_6$, this corresponds to the large-m limit, with n and the interaction constant $k = c_6 g_s N l_s$ held fixed. We see that in this limit the energy and the Bohr radius of the bound states scales like

$$E \propto \frac{m^{1/3}}{k^{2/3}}\,, \qquad a_B \propto \frac{k^{1/3}}{m^{2/3}}\,. \tag{2.63}$$

We see from this that the quantum bound states are not reliable in the limit of large V_6, for fixed N, because in this limit they imply $a_B \ll l_s$.

2.5 Compactification

We have found potentially long-lived orbits for probe antibranes moving in the field of N source branes, provided $\tilde{d} = 1$ and so they move in 3 transverse dimensions. For 10 dimensions overall, this is what singles out D6-branes for special attention. For later applications to brane-world cosmology we will be interested in branes whose world-volume is 4 dimensional, and so we pause here to consider how this may be achieved from the above solutions by compactification.

2.5.1 Toroidal reduction

Our goal in this section is to dimensionally reduce the action, eq. (2.1), from D to $D' = D-1$ dimensions. The point is to present a consistent lower-dimensional truncation of this action, in the sense that any solution of the lower-dimensional equations of motion are also solutions of the equations of the full system, eq. (2.1), following the construction of ref. [10].

To this end imagine that one of the coordinate directions, denoted z, is periodic. In this case we write the D-dimensional metric, ds^2_D, in terms of a D'-dimensional metric, $ds^2_{D'}$ plus a scalar field, φ, and a vector field, $\mathcal{B}_M$ as follows:

$$ds^2_D = e^{2\hat{a}\varphi}\, ds^2_{D'} + e^{2\hat{b}\varphi}\,(dz + \mathcal{B}_M dx^M)^2\,, \tag{2.64}$$

with $\hat{b} = -(D'-2)\,\hat{a}$ and $\hat{a}^2 = 1/[2(D'-1)(D'-2)]$ chosen to ensure canonical Einstein and φ kinetic terms. The D-dimensional $(n-1)$-form field, $A_{[n-1]}$, is similarly decomposed into D'-dimensional $(n-1)$- and $(n-2)$-form fields, $B_{[n-1]}$ and $B_{[n-2]}$, according to

$$A_{[n-1]} = B_{[n-1]} + B_{[n-2]} \wedge dz\,. \tag{2.65}$$

With these choices the action, eq. (2.1), dimensionally reduces to

$$\begin{aligned} S_{D'} \;=\; -\int d^{D'}x\,\sqrt{-g_{D'}}\,\Bigg[&\frac{1}{2}\,R_{D'} + \frac{1}{2}\,(\partial\phi)^2 + \frac{1}{2}\,(\partial\varphi)^2 + \frac{1}{4}\,e^{-2(D'-1)\hat{a}\varphi}\,\mathcal{B}^2_{[2]} + \\ &+ \frac{1}{2n!}\,e^{-2(n-1)\hat{a}\varphi+\alpha\,\phi}G'^2_{[n]} - \frac{1}{2(n-1)!}\,e^{2(D'-n)\hat{a}\varphi+\alpha\,\phi}\,G^2_{[n-1]}\Bigg]\,, \end{aligned} \tag{2.66}$$

where the field strengths appearing in this expression are defined by

$$\mathcal{B}_{[2]} = d\mathcal{B}, \qquad G_{[n-1]} = dB_{[n-2]}, \qquad G'_{[n]} = dB_{[n-1]} - G_{[n-1]} \wedge \mathcal{B}\,. \tag{2.67}$$

This lower-dimensional action may be consistently truncated in three distinct ways, in the above sense of consistency. The three ways are obtained by setting to zero all fields except the D'-dimensional metric and one of the following pairs of form-fields and canonically-normalized scalars [10]:

$$\begin{aligned} &\text{Option A:} && \mathcal{B} \quad\text{and}\quad \phi'_1 = \varphi \\ &\text{Option B:} && B_{[n-2]} \quad\text{and}\quad \phi'_{n-2} = \frac{2(D'-n)\,\hat{a}\varphi + \alpha\phi}{\sqrt{4(D'-n)^2\,\hat{a}^2 + \alpha^2}} \\ &\text{Option C:} && B_{[n-1]} \quad\text{and}\quad \phi'_{n-1} = \frac{-2(n-1)\,\hat{a} + \alpha\phi}{\sqrt{4(n-1)^2\,\hat{a}^2 + \alpha^2}}\,. \end{aligned}$$

Each of these reductions produces a D'-dimensional action which again has the form of eq. (2.1), but with parameter $\alpha = \alpha_{D'}$ now given in terms of the higher-dimensional parameter α_D by:

$$
\begin{aligned}
&\text{Option A:} \qquad && \alpha_{D'} = -2(D'-1)\alpha_D \qquad && \text{implying} \qquad \Delta' = 4\,,\\
&\text{Option B:} \qquad && \alpha_{D'} = \sqrt{4(D'-n)^2\hat{a}^2 + \alpha_D^2} \qquad && \text{implying} \qquad \Delta' = \Delta\,,\\
&\text{Option C:} \qquad && \alpha_{D'} = \sqrt{4(n-1)^2\hat{a}^2 + \alpha_D^2} \qquad && \text{implying} \qquad \Delta' = \Delta\,.
\end{aligned}
\tag{2.68}
$$

The two main choices are now whether to apply this reduction to the case where the compactified dimensions are transverse or parallel to the source and probe branes' world volumes, and so we now consider both of these options.

2.5.2 Parallel dimensions

The simplest option is to compactify dimensions which are parallel to the branes' world volumes by wrapping the branes about a torus. Wrapping D6-branes about a 3-torus of radius R in this way would produce a brane world which is effectively 4 dimensional when viewed on scales large compared with R by virtue of the standard Kaluza-Klein mechanism. Such a toroidal compactification is straightforward since the solutions we have considered are invariant under translations along the directions parallel to the branes. In particular, it is clear that the solution, eq. (2.3), remains a solution to the full higher-dimensional field equations if some of the coordinates x^μ are required to be periodic.

Writing $x^p = z$ and directly comparing eqs. (2.3) with the Kaluza-Klein ansätze, eqs. (2.64) and (2.65), allows the field configurations to be written purely in terms of lower-dimensional fields as follows:

$$
ds^2_{D'} = h^{-\tilde{\gamma}'}\, dx^2_{p'} + h^{\gamma'}\, dy^2\,, \quad e^{\phi} = h^{\kappa}\,, \quad e^{\varphi} = h^{\rho}\,, \quad (B_{[n-2]})_{01\ldots p-1} = \zeta(1-h^{-1})\,, \tag{2.69}
$$

where κ and ζ are as before (c.f. eq. (2.4)) and

$$
\rho = \frac{\tilde{\gamma}}{2(D'-2)\,\hat{a}}, \quad \gamma' = \gamma - 2\,\hat{a}\,\rho, \quad \tilde{\gamma}' = \tilde{\gamma} + 2\,\hat{a}\,\rho. \tag{2.70}
$$

Notice that although this solution resembles truncation option B of eq. (2.68), it is *not* simply a special case of the solutions we have considered so far, with the replacements $D \to D'$, $p \to p'$ and $\alpha \to \alpha'$. It is not because in this solution *both* of the scalar fields ϕ and φ vary nontrivially in the transverse dimensions. (In particular, the combination orthogonal to ϕ'_{n-2} of eq. (2.68) does not vanish.)

By starting from a given p-brane solution in D dimensions and successively compactifying k of the parallel dimensions we obtain new configurations which describe $p-k$ branes moving in $D-k$ dimensions, without changing the number of transverse dimensions and so also not changing $\tilde{d}$. (These are the 'diagonal' reductions of ref. [10].) By construction this reduction leaves the values of the parameters η, ω or ζ which appear in the probe brane action unchanged.

Reducing the BPS Dp-branes in 10 dimensions in this way then leads to a family of $p-k$ branes moving in $10-k$ dimensions, for which $\eta = \omega = \zeta = 1$, and so for which the orbital properties are particularly simple. The choices $p = 6$ and $k = 3$ give the particularly interesting case of 3-branes moving in 3 transverse dimensions. For these $\tilde{d} = 1$ and so the orbits are the closed ellipses described earlier.

2.5.3 Transverse dimensions

To this point we have treated the transverse dimensions as if they were infinite in extent. Although this type of geometry might have brane-world applications if gravitons can be confined to our brane (perhaps through the warping of spacetime), we need not rely on this when exploring the various implications of branonium. If not, we must also address the dimensional reduction of the transverse dimensions.

Reduction of the transverse dimensions is trickier because the field configurations in these directions are not translationally invariant. For BPS branes, such as the 10-dimensional Dp-branes, this can be dealt with using a generalization of the solutions we have considered to source branes which are not all located at the same point. This more general solution is obtained using the same expressions, eqs. (2.3), as before but with the harmonic function of eq. (2.6) replaced by the new expression

$$h(y) = 1 + \sum_i \frac{k}{|\mathbf{y} - \mathbf{y}_i|^{\tilde{d}}}\,, \tag{2.71}$$

where $|\mathbf{y} - \mathbf{y}_i|^2 = \delta_{mn}(y-y_i)^m(y-y_i)^n$. Given this solution a toroidal compactification of the transverse dimensions may be performed simply by including the appropriate image charges for the source branes, which serve to ensure that the transverse dimensions become periodic [10].

There are two cases of this kind of reduction which concern us, depending on the relative size of the compact dimensions and the size of the brane orbits which are of interest. If some of the transverse dimensions are very small compared to the orbital sizes of interest, then we may integrate out these directions and work in the limit where their radius is taken to zero. If k transverse directions are so reduced it has the effect of reducing $D \to D' = D - k$ without changing d, and so gives new brane solutions for which $\tilde{d} \to \tilde{d}' = \tilde{d} - k$. (These are the 'vertical' reductions of ref. [10].)

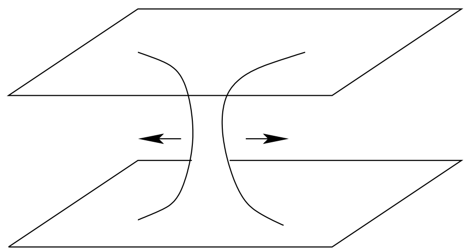

Figure 3: Nucleation of a throat between a brane and and antibrane.

For the brane orbits of previous sections we must take $\tilde{d} > 2$, however, and so at least some of the transverse dimensions must have radii, R, which are larger than the orbital sizes, a, of interest. In this case the orbital analysis of the previous sections must be considered to be only approximately valid, up to corrections which are of order $(a/R)^{\tilde{d}}$ for $a \ll R$. For $a \sim R$ the influence of the image sources can significantly alter the probe-brane motion. In our later applications we imagine at least three of the transverse dimensions to be stabilized with sufficiently large radii.

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3. Stability

We now turn to the stability of the orbits described in the previous section (see [15] for a related discussion). There are several kinds of potential instability to consider. To this point we have treated the probe-brane motion as if the branes were perfectly rigid, with only their centres of mass free to move. The first instability issue to address is the extent to which this is a good approximation, inasmuch as other kinds of brane motion might become excited by the movement of the centre of mass. To address this at the classical level we look for tachyonic modes among the linearized fluctuations of the brane position, $y(\xi, t)$, about the field configurations of interest. We argue that this kind of instability need not be present so long as the source and probe branes are kept at distances which are much larger then the string length.

The second stability issue asks whether the brane motion produces radiation of massless bulk and/or brane fields, thereby radiating away energy and so causing the brane orbits to decay. This kind of instability does occur, and we here estimate how long it takes to destroy a given brane orbit.

All told we identify three separate kinds of instabilities to which orbiting branes are subject.

- Brane annihilation, due to the perturbative open string tachyon. If the brane and antibrane get within a string length or so of one another, then they can annihilate with the antibrane being absorbed by the N source branes, ultimately leaving a configuration of $N-1$ branes. The energy released is then radiated into modes in the bulk and on the remaining source branes. We do not pursue this in detail because our present interest is in branes which are sufficiently well separated as to avoid this. Notice that, depending on their number and size, for compact transverse dimensions sufficient separation of the branes can require them not to be too differently wound around the extra dimensions.

- A non-perturbative brane decay is possible, à la Callan-Maldacena [13]. These are solutions to the Euclidean equations of motion. They can be interpreted as sphalerons associated to the decay of a brane-antibrane system. A throat is nucleated that joins the brane-antibrane pair. Then it expands and transforms the brane world-volume into kinetic energy of the expansion front (see figure 3). We do not pursue this instability further since its nucleation probability is proportional to e^{-1/g_s}, and so is exponentially small for small string coupling.

- Radiation into bulk modes, such as the graviton and dilaton, or brane modes can occur and arises at low orders in the string coupling. We expect this to be the dominant instability for large interbrane separations and weak couplings, and estimate the time for orbital decay below.

We will then establish first the stability of the system against bending and concentrate later only on the instability due to radiation.

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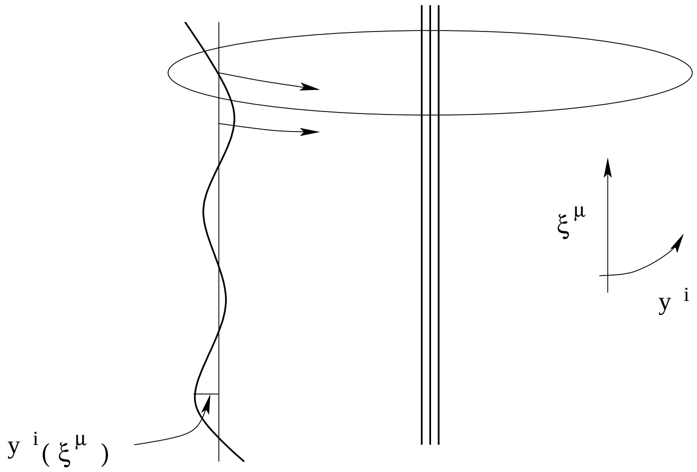

Figure 4: The fields $y^i(\xi)$ on the brane parameterise its fluctuations. Instabilities corresponding to brane bending or the classical orbits can be analyzed from the mass of these modes in the quadratic approximation.

3.1 Stability against brane bending

Instead of considering only explicitly straight and parallel branes, imagine allowing them to bend so that the probe-brane position relative to the centre-of mass is $y^i(t, \xi^\mu) = y^i - y^i_0$, where as before $\{t, \xi^\mu\}$ are the coordinates along the brane (see figure 4). Nontrivial dependence of y^i on ξ^μ corresponds to brane bending, while perturbations depending only on time, i.e. $y^i := y^i(t)$, describe the rigid motion of the brane centre of mass.

A stability analysis is performed by expanding the action about a particular background solution, $y^i = y^i_0(t) + \delta y^i(\xi, t)$, with the background $y_0(t)$ taken to describe the rigid brane motion described in the previous sections. The expansion of L about this solution defines the system's *routhian* [12], which to quadratic order is

$$L_{\rm eff} = -R = \frac{1}{2} A^{\mu\nu}_{ij} \; \partial_\mu \delta y^i \, \partial_\nu \delta y^j + B^\mu_{ij} \; \delta y^i \, \partial_\mu \delta y^j + \frac{1}{2} C_{ij} \; \delta y^i \delta y^j \,, \tag{3.1}$$

where

$$C_{ij}[y_0(t)] = \left. \frac{\partial^2 R}{\partial y^i \, \partial y^j} \right|_{y^0} , \qquad B^\mu_{ij}[y_0(t)] = \left. \frac{\partial^2 R}{\partial(\partial_\mu y^i) \, \partial y^j} \right|_{y^0} \tag{3.2}$$

and

$$A^{\mu\nu}_{ij}[y_0(t)] = \left. \frac{\partial^2 R}{\partial(\partial_\mu y^i) \, \partial(\partial_\nu y^j)} \right|_{y^0} . \tag{3.3}$$

One must analyse this effective lagrangian looking for unstable perturbations, given the known functions for the quantities $A^{\mu\nu}_{ij}$, B^μ_{ij} and C_{ij}.

For the present instance, we may for simplicity focus on the large-separation limit and so take the non-relativistic approximation

$$L \approx m \left[\frac{v^2}{2} + (\eta - \hat{q}) \frac{k}{r^{\tilde{d}}} \right] . \tag{3.4}$$

The stability analysis greatly simplifies in this case because simplicity of the derivative terms implies the quantities $A^{\mu\nu}_{ij}$ and B^{μ}_{ij} are independent of fields, and so are also independent of time. As a result it also follows that $B^{\mu}_{ij} = 0$.

The simplicity of the kinetic terms allows a further simplification, since it implies gradients contribute a strictly positive amount to the energy density. This means that it is the zero-momentum modes in a Fourier decomposition, $\delta y^i(k) \propto e^{i\vec{k}\cdot\vec{\xi}}$, which are the most unstable. It follows that if these zero-momentum modes are stable, then so must be all of the modes having nonzero momentum. This stability of nonzero momentum modes has a simple physical interpretation. On one hand, if the probe brane were to bend slightly towards the source branes, then tidal forces from the source branes will act to increase the amplitude of this bending. The amount of this tidal disturbance will fall with increasing brane separation. On the other hand, the brane tension resists all such bending and easily wins so long as the interbrane separation is large compared with the string length.

We now specialize to the stability analysis for the zero-momentum modes in a purely time-dependent background. The effective lagrangian for the perturbations $\delta y^i = y^i - y^i_0$ then simplifies to

$$L_{\rm eff} = -R = \frac{1}{2}\mathcal{A}_{ij}\delta\dot{y}^i\delta\dot{y}^j + \mathcal{B}_{ij}\delta\dot{y}^i\delta y^j + \frac{1}{2}C_{ij}\delta y^i\delta y^j \,, \tag{3.5}$$

where $\mathcal{C}_{ij} = \frac{\partial^2 R}{\partial y^i\,\partial y^j}|_{y_0}$, $\mathcal{B}_{ij} = \frac{\partial^2 R}{\partial \dot{y}^i\,\partial y^j}|_{y_0}$ and $\mathcal{A}_{ij} = \frac{\partial^2 R}{\partial \dot{y}^i\partial \dot{y}^j}|_{y_0}$. Applied to the lagrangian, eq. (3.4), we get the following routhian expansion for the perturbations:

$$\frac{L_{\rm eff}}{m} = -R \approx \frac{1}{2}\Big[(\delta\dot{r})^2 + r_0^2\,(\delta\dot{\theta})^2 + (\delta\dot{y}^i)^2\Big] + \frac{1}{2}\left[(\eta-\hat{q})\frac{k\tilde{d}(\tilde{d}+1)}{r_0^{\tilde{d}+2}} - 3\left(\frac{\ell}{r_0^2}\right)^2\right](\delta r)^2 \,. \tag{3.6}$$

Only the radial mode appears nontrivially in these manipulations, leading to the 'mass':

$$M_r^2 = -m(\eta-\hat{q})\frac{k\tilde{d}(\tilde{d}+1)}{r^{\tilde{d}+2}} + \frac{3\ell^2}{r^4} \,. \tag{3.7}$$

Our interest is in $\eta - \hat{q} > 0$, such as occurs for probe Dp-antibranes, and so the first term in M_r^2 is negative while the second term is positive. Which term dominates at large r depends on the relative size of the power of r in the two terms, and so $M_r^2 < 0$ for large r when $\tilde{d} < 2$ and $M_r^2 > 0$ for large r when $\tilde{d} > 2$. In the intermediate case $\tilde{d} = 2$ the sign of M_r^2 at large r is the same as the sign of the combination $3\ell^2 - mk(\eta-\hat{q})\,\tilde{d}(\tilde{d}+1)$. For the D$p$-branes this means that we have classical instability at large r for $p > 5$ and classical stability for $p = 6$. This is as expected since we know that deviations from the classical orbit in the Kepler problem for $\tilde{d} = 1$ are purely oscillatory.

Since D6-branes can be supersymmetric, the oscillations about the stable configuration fill out an entire supermultiplet of the appropriate unbroken supersymmetries. Seen from 4 dimensions on the brane, this could be an $N = 4$ SYM multiplet, depending on how many supersymmetries are broken by the compactification.

A similar analysis can be done to study the stability of the brane/antibrane system against developing a relative rotation as their centres of mass move. This does not occur because parallel branes furnish a local minimum of the potential energy as a function of angle [11]. For branes embedded in infinite 10D spacetime, the oscillation time scale for small rotational perturbations can be quite large, due to the necessity to rotate an extremely long brane. In this case one should check how this oscillation time scale compares with the orbital period calculated above. For real applications this time scale is less important, however, because the branes then move within compact transverse dimensions. For toroidal compactifications, for instance, branes cannot rigidly rotate relative to one another without a large energy cost. This cost arises because local rotation of the branes necessarily requires the branes to bend somewhere, due to the necessity that they cannot change their windings about the cycles of the torus.

We do not expect this result to change when relativistic corrections are taking into account, and so we expect the relativistic system to also be stable (for $\tilde{d} = 1$) provided the interbrane separation is kept well away from the string length. We now turn to the orbital energy loss due to radiation of various string modes.

3.2 Radiation into the bulk

The most important decay mode for large orbits at weak coupling is energy loss through radiation into the various massless fields in the problem. For the classical motion which we are considering, this radiation causes the orbits to slowly shrink in size, much as the binary-pulsar orbit has been observed to do due to gravitational radiation. Once the orbital size reaches the string scale the much quicker open-string-tachyon instability takes over, causing a catastrophic annihilation of the antibrane with the brane. Our purpose in this section is to estimate the time which is required before this catastrophe takes place.

3.2.1 Radiation rates

Unlike the situation for the binary pulsar, the moving probe antibrane has several types of charges which couple its motion to fields whose radiation could carry away energy, including at the very least the bulk gravitational, $(n-1)$-form and dilatonic fields. For nonrelativistic motion, when all other things are equal, the main carriers of radiated energy are the lower-spin fields [17].[2] This is because in this limit spin-0, -1 and -2 particles respectively dominantly couple to the orbiting brane's monopole-, dipole- and quadrupole-moments, with each higher multipole suppressed by additional small factors of powers of $1/r$. We therefore base our estimate on radiation into a bulk scalar field like ϕ.

Suppose there are D dimensions which are large compared with the brane orbits, of which d are parallel to the various branes. The case of special interest is $d = 4$ and $\tilde{d} = 1$, which corresponds to $D = 7$, but for the present purposes we need not restrict ourselves to this case. We imagine the probe brane orbit to be circular, with orbital radius r and orbital frequency Ω.

Recall that ϕ appears in the brane action through the overall factor $e^{-\phi}\, m$, with (as usual) $m = T_p\, V_p$, where the spatial volume of the brane enters because of the overall translation invariance within the brane dimensions. This action is given in units where

[2] This observation also applies to the binary pulsar in the presence of very light scalar and vector particles, as may be seen, for example, in ref. [16].

the scalar and Einstein kinetic terms are $S = -\frac{1}{2}\int d^D x \sqrt{-g}\,[R + (\partial\phi)^2]$, and so we see that the ϕ-brane coupling is proportional to $\kappa_T\, m$, where $\kappa_T^2 = 8\pi G_D/V_p$ relates κ_T to the D-dimensional Newton's constant. Furthermore, since for slowly-moving objects Lorentz-invariance precludes any radiation unless the source accelerates, the ϕ-emission amplitude should be proportional to the brane acceleration, a_b.

Keeping track of dimensions and factors of 2π, the power radiated into scalars per unit brane volume therefore has the schematic form

$$P \sim 2\pi\, \kappa_T^2\, m^2\, a_b^2 \int \delta(E - \Omega) \frac{d^{D-d}k}{(2\pi)^{D-d}\, E^2}\,, \tag{3.8}$$

where the final integral is over the phase space of the emitted scalar. The Dirac δ-function is meant to express the fact that energy conservation demands the dominant energy of the outgoing scalars to be set by the dominant frequency, Ω, of the time-dependent source. It is this energy which sets the scale of the phase space integrations. The power of E in the integral is chosen on dimensional grounds.

We estimate the phase-space integration by replacing it with the appropriate power of the relevant energy scale, Ω, and by counting the factors of 2π which are produced by the angular integrations in the transverse dimensions. This leads to the estimate that the phase-space integral is of order $(2\pi)^{[(D-d)/2]}\,\Omega^{D-d-3}$, where $[k/2]$ denotes the integer part of $k/2$. We find in this way the following estimate for the total power radiated:

$$P \sim \kappa_T^2\, m^2\, a_b^2 \left(\frac{\Omega^{D-d-3}}{(2\pi)^{\mathcal{P}}}\right), \tag{3.9}$$

with $\mathcal{P} = D - d - 1 - [\frac{1}{2}(D-d)]$.

For three transverse dimensions we would have $D - d = 3$ and so $[\frac{1}{2}(D-d)] = 1$. This would be appropriate for point particles (0-branes) moving in 4 dimensions, in which case we would have $D = 4$ and $d = 1$ and $\kappa_T = \kappa_4$ is the ordinary 4D gravitational coupling. The above estimate then reproduces the results of more careful calculations, which give:

$$P = \frac{1}{8\pi}\,(\kappa_4 m)^2 a_b^2\,. \tag{3.10}$$

This same formula applies equally well to the case of more general branes orbiting in 3 transverse dimensions, provided we use the appropriate choice for κ_T

$$P = \frac{1}{8\pi}\,\kappa_T^2\, m^2\, a_b^2 \;= \frac{1}{8\pi}\,\kappa_D^2\, T_p^2\, a_b^2\, V_p\,. \tag{3.11}$$

Notice that in all cases P scales with the spatial volume of the orbiting branes, as is required by their translational symmetry, and so the power radiated per unit energy is independent of V_p. As a consequence of this V_p drops out of the time required for an orbit to decay, as we shall shortly see in more detail. Also of note is the dramatic phase-space suppression (by powers of $\Omega\, l_s$) which arises when there are more than 3 transverse dimensions.

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3.2.2 Orbital decay time

We now turn to the time required for a circular orbit of radius r to decay, given the above estimate for the power radiated. We focus on the case $\tilde{d} = 1$, since it is for this case that these stable orbits are most likely to exist. For the present purposes we write the power radiated as $P = K\, m^2\, a_b^2$, with K containing the various dimension-dependent constants.

To evaluate the acceleration it suffices to work in the limit of large orbits and small velocities, where the brane lagrangian becomes (up to an additive constant)

$$L = \frac{mv^2}{2} - V(r)\,, \tag{3.12}$$

where $V = -2k\,m/r$. The acceleration is then

$$a_b^2 = \frac{1}{m^2}\left(\frac{\partial V}{\partial r}\right)^2 = \frac{4k^2}{r^4}\,, \tag{3.13}$$

and so the power radiated becomes

$$P = -\frac{dE}{dt} = Km^2\,\frac{4k^2}{r^4} = \frac{4\,K\,E^4}{m^2 k^2}\,. \tag{3.14}$$

This last equality uses the nonrelativistic expression which relates the energy and radius of a circular orbit: $E = -2\,mk/(2\,r) = -mk/r$.

The time taken for the radius to change from r_i to r_f is therefore

$$\tau = -\int_{E_i}^{E_f} \frac{dE}{P} = \frac{m^2k^2}{4\,K}\int_{E_f}^{E_i}\frac{dE}{E^4} = \frac{(r_i^3 - r_f^3)}{12\,mk\,K}\,. \tag{3.15}$$

Taking $r_f \sim l_s \ll r_i$ and comparing τ with the initial orbital period, $T = 2\pi\sqrt{r_i^3/2k}$ gives an estimate of the number of orbits traversed before decay:

$$\mathcal{N} = \frac{\tau}{T} = \frac{r_i^{3/2}}{12\pi\sqrt{2k}\,mK}\,. \tag{3.16}$$

To see how these results scale with the microscopic quantities, we may use the D6-brane results $m = T_6V_6 = V_6/((2\pi)^6\, {l_s}^7 g_s)$ and $k = c_6 g_s N l_s = g_s N l_s/2$ and $K = \kappa_{10}^2/(8\pi\, V_6)$ with $\kappa_{10}^2 = (2\pi)^6\pi g_s^2 {l_s}^8$. Using these results the number of orbits before decay becomes

$$\mathcal{N} = \frac{2}{3\pi}\left(\frac{r_i}{l_s}\right)^{3/2}\frac{1}{(g_s^3\, N)^{1/2}}\,. \tag{3.17}$$

This dependence is relatively easy to understand.

- As already discussed, V_6 does not appear in eq. (3.17) because P and E are both proportional to the brane volume.
- The string coupling dependence arises from several sources. First notice that the orbital energy does not depend on g_s because the contribution cancels between the

interaction constant k and the tension T_p. For a similar reason g_s drops out of the combination $m^2\kappa^2$ in the radiation rate. The entire contribution of g_s to the radiation rate therefore arises through the acceleration, which vanishes if $g_s \to 0$. Taken all together this gives a $1/g_s^2$ dependence to the lifetime, which combines with the $g_s^{1/2}$ arising from the $k^{1/2}$ in the orbital period, to make $\mathcal{N} \propto g_s^{-3/2}$.

- The N factor is the relative charge of the set of branes to the antibrane. The bigger is N, the stronger is the acceleration of the probe brane and so the shorter is the branonium lifetime.

We see that the branonium lifetime is maximized the smaller we take $g_s^3\, N$ and the larger we take the orbit to be. As discussed in previous sections, if we wish to use the full nonlinear field equations rather than just their linearized form, consistency requires $g_s\, N \gg 1$, and so a long-lived orbit would require g_s and r/l_s to be chosen that much larger.

Of course if the brane were in the quantum limit, this orbital decay instability could be prevented just as is done for electromagnetic decay in atoms. Unfortunately this mechanism is not available to us for branes in the limit of large V_p, since for these the stabilization occurs at distances of order the Bohr radius from the source branes. As we have seen earlier, in the decompactification limit the stabilizing bound states lie beyond the domain of validity of our approximations, since their size is smaller than the string length. It might nonetheless be of interest in some contexts, such as in the very early universe if we assume that all dimensions start with a size close to the string scale.

3.3 Radiation into brane modes

We have seen that probe brane orbits decay due to radiation into bulk modes, and but this radiation could also be accompanied by radiation onto the brane. We next perform an estimate of this rate, along the lines of the one done earlier for bulk radiation, with the conclusion that radiation into brane modes is subdominant to bulk radiation.

We use for these purposes the coupling of the brane-bound gauge field, given by the (String-Frame) Born-Infeld action:

$$S_{\rm BI} = -T_p \int d^d\xi \; e^{-\phi} \sqrt{-\det(\hat\gamma_{MN})\; \left[1 + {l_s}^4 \hat\gamma^{\mu\lambda}\hat\gamma^{\nu\rho}\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho} + \cdots\right]}\,, \tag{3.18}$$

where $\hat\gamma_{\mu\nu} = \hat g_{MN}\partial_\mu x^M \partial_\nu x^N$ is the brane's induced metric and $\mathcal{F}_{\mu\nu}$ is the gauge field strength. We see that the relevant coupling is in this case $\kappa_B = T_p\, {l_s}^4\, e^{-\phi_b}$, where $\phi_b = \phi(y_b)$ is the dilaton field evaluated at the position of the brane.

Although radiation into the bulk requires non-vanishing acceleration, this is not sufficient to produce radiation into brane modes. This is because brane radiation arises due to the time dependence of the fields which probe-brane-bound observers see as a result of their motion through the fields of source branes (much as occurs in mirage cosmology). Since this time-dependence only arises if the interbrane distance, r, changes with time, brane radiation does *not* occur for circular brane orbits.

The role of a_b^2 in the bulk radiation rate is in this case therefore played by $(dh/dt)^2 \sim (2k\,\dot{r}/r^2)^2 \sim (2k\,\Omega\,e/r)^2$, where e is the orbital eccentricity (which we take to be small). Keeping in mind that translation invariance requires the power radiated to be proportional to V_p and that the particles be produced in pairs — with opposite momenta to conserve momentum — we obtain the following estimate for the power radiated per unit spatial volume

$$\frac{P_p}{V_p} \sim 2\pi\,\kappa_B^2 \left(\frac{dh}{dt}\right)^2 \int \delta(E_1 + E_2 - \Omega)\,\delta^p(k_1 + k_2)\,\frac{d^pk_1\,d^pk_2}{(2\pi)^{2p}\,E^{2p-7}} \sim \frac{\kappa_B^2\,\dot{h}^2\,\Omega^{6-p}}{(2\pi)^{\mathcal{P}}}\,, \tag{3.19}$$

where the power of energy is included on dimensional grounds and $\mathcal{P} = 2p - 1 - [p/2]$.

Choosing 3-branes, and using $\kappa_B = T_4\,{l_s}^4\,e^{-\phi_b} = 1/g_b$, where $g_b = g_s e^{\phi_b}$ is the string coupling at the position of the brane, gives $[p/2] = 1$ and so

$$\frac{P_3}{V_3} \sim \frac{\Omega^3\,\dot{h}^2}{g_b^2(2\pi)^4} \sim \frac{e^2\,\Omega^5}{g_b^2(2\pi)^4}\left(\frac{2k}{r}\right)^2 \sim \frac{e^2}{(2\pi)^4\,g_s^2\,r^5}\left(\frac{2k}{r}\right)^{9/2} \sim \frac{e^2}{(2\pi)^4\,g_s^2\,r^5}\left(\frac{g_sN\,l_s}{r}\right)^{9/2}. \tag{3.20}$$

Here we use the large-r results $\Omega = \sqrt{2k/r^3}$, $g_b = g_s[1 - (2k\kappa/r) + \cdots]$, as well as the relation $2\,k \sim g_sN\,l_s$. The dependence of this estimate on Ω, e, and k/r is borne out by a toy calculation of the rate of change of the vacuum energy density on the brane which is given in the appendix.

Several conclusions may be drawn from this estimate:

- Comparing the power radiated into the bulk (3.11) and (3.20) gives the ratio of power radiated into the bulk and to the brane. For D3-branes moving in 3 transverse dimensions this ratio becomes:

$$\frac{P_{\rm brane}}{P_{\rm bulk}} \sim \frac{e^2\,\Omega^3}{g_s^2(2\pi)^3\kappa_7^2\,T_4^2\,a_b^2}\left(\frac{dh}{dt}\right)^2 \sim \frac{e^2\,{l_s}^3}{g_s^2(2\pi)^3\,r^3}\left(\frac{2k}{r}\right)^{5/2} \sim \frac{e^2(g_sN)^{5/2}}{(2\pi)^3 g_s^2}\left(\frac{l_s}{r}\right)^{11/2}. \tag{3.21}$$

  This shows that the greater part of the emitted radiation goes into the bulk.

- Although most of the radiation goes into the bulk, particles *are* produced on the branes, and this has possible implications for cosmology. Provided the number of source branes differs from the number of probe branes, some branes survive even after the orbit eventually decays. If, for instance, we imagine ourselves being trapped on one of these remaining branes, these particles can equilibrate with observable particles at later times. Depending on the details, this could provide a source of reheating and/or baryogenesis. For a branonium-inflation scenario these produced particles are likely to be inflated away as they are produced, and so are unlikely to obviate the necessity for other sources of reheating, such as due to the brane-antibrane annihilation itself.

4. Conclusions

We have seen that the bound states of brane systems have very interesting properties. The present work should be considered only as the beginning of their exploration. Within a brane world context we find it appealing to imagine our entire universe as being, albeit briefly, a member of a 'solar system' of universes.

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Our analysis has included only branes and antibranes which share the same dimensionality and are parallel, which makes a closer analogy with positronium than with a stable atom. It is straightforward to generalise our discussion to a system of branes having different dimensionality and/or branes at angles, for which the effective interactions can still be attractive. We would expect the behaviour and instability to annihilation of these systems to be very similar, although a detailed analysis is left for the future. Similarly one might extend this idea to possible bound states between branes and orientifold planes. Other systems, including lifts to 11 dimensions and/or T- and S- dual versions might also be considered.

Many open questions remain. One is a more fundamental understanding of why D-brane/antibrane orbits do not precess, having the same shape for fully relativistic systems as they have in the non-relativistic limit. There may exist a mapping between the two lagrangians in a similar way as recently found in ref. [18] for the case of a massive particle near the horizon of an extreme Reisner-Nordström black hole. Notice that Bertrand's theorem [12] states that the only systems giving rise to closed orbits in the central force problem are the $1/r$ potential and the harmonic oscillator. The D-brane systems we consider here are not counterexamples to this theorem because of the canonical kinetic terms which the proof of Bertrand's theorem assumes. It would be interesting to see how other classical results, such as the virial theorem, the three- (or more) body problem or chaotic orbits, generalize in the present instance.

Our quantum treatment of the branonium system was very basic. A better understanding of how to properly describe the system at the quantum level and the physical interpretation of the corresponding wave function would be desirable, perhaps with implications to quantum cosmology. It would also be interesting to know in more detail the back reaction of the probe brane on the bulk metric, both to better understand the $N = 1$ brane/antibrane problem and to have an explicit microscopic derivation of the long-distance fields which are of interest for cosmological applications. Because the branes behave as point masses in the transverse dimensions it is clear that for 3 or more transverse dimensions the bulk gravitational (and other) fields in the extra dimensions can be made negligible simply by keeping them well separated. Experience with co-dimension 1 branes can be misleading in this case, because the bulk fields which these give rise to can instead be homogeneous spacetimes like de Sitter or anti-de Sitter space.

Finally, the applications to cosmology are potentially many. Inflation can be analyzed in a way similar to ref. [2]. In this case the system has several novel features. For instance it provides an unusual slow roll because it is the angular variable, θ, which rolls even though the potential is exactly flat in this direction. Nonetheless, the constant inflationary energy density can be expected to evolve as the orbital radius changes due to energy loss into bulk radiation. This changes the analysis in an interesting way, since $\dot{r}$ is not related to dV/dr in the usual expression. Furthermore, the fact that the orbit is unstable and decays is likely to make the brane motion appear as a very unusual 'fluid' from the cosmological point of view. As for other forms of brane/antibrane inflation, the end of inflation would occur once the antibrane falls into the source branes and annihilates, providing the usual extremely rich realization of hybrid inflation.

The orbital motion of the probe brane might also provide other novel cosmologies, such as a particularly interesting example of a mirage cosmology [19]. For elliptical orbits the periodic increase and decrease of the interbrane separation gives rise to a cyclic cosmology for the trapped brane observers. Because the small size of the extra dimensions implies the period of these cycles is microscopically short for standard compactifications, this is likely to be relevant only for very early cosmology. The particle production which is produced by the time dependence of the induced fields on the branes in these cosmologies may have implications, perhaps providing a geometric realization of the Affleck-Dine mechanism for baryogenesis. Because these speculations take us considerably beyond the scope of the present paper, we devote a companion article [20] to a more detailed discussion of these cosmological issues.

A. Quantum energy on the brane

We here briefly record a toy calculation of the time dependence of the quantum stress energy density on the brane which arises due to the time dependence of the induced brane metric.

Our toy model consists of a conformally coupled scalar, in 4 spacetime dimensions, defined by the action

$$\frac{\mathcal{L}_{\rm toy}}{\sqrt{-g}} = -\,\frac{1}{2}\,(\partial\varphi)^2 - \frac{1}{6}\,R\varphi^2\,. \tag{A.1}$$

We evaluate the quantum stress energy tensor for this scalar using a time-dependent but conformally-flat metric

$$ds^2 = h^{-1/2}(-dt^2 + d\xi^2)\,, \tag{A.2}$$

where $h = 1 + k/r$ and $r = r_0\,[1 + e\,\cos(\Omega t)]$. Here e and Ω represent the eccentricity and the angular frequency of the radial oscillations for a slightly eccentric elliptical orbit.

Now, for any theory which is conformally trivial, the expectation value of the stress energy tensor, $T_{\mu\nu}$, is determined by integrating the conformal anomaly, leading to the expression [21]

$$\begin{aligned}\langle T_{\mu\nu}\rangle \;=\; \frac{1}{1440\pi^2}\Bigg[& aR_{;\mu\nu} - a g_{\mu\nu}\Box R + \left(a - \frac{b}{3}\right) RR_{\mu\nu} + \\ & + \left(-\frac{a}{4} + \frac{b}{8}\right) g_{\mu\nu}R^2 + \frac{b}{2}R_{\mu\alpha}R^{\alpha}_{\nu} - \frac{b}{4}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\Bigg]\,.\end{aligned} \tag{A.3}$$

Here the constants a and b depend on the spin of the field in question. For a conformal scalar fields we have $a = -1/6$ and $b = 1$, while for a massless vector field we have, $a = -3$ and $b = 62$.

Inserting the metric, eq. (A.2), into eq. (A.3), making use of the relations $\omega = \sqrt{2k/r^3}$, and $m = T_pV_p$ we find (to leading order in $1/r_0$ and e):

$$\rho = \langle T_{00}\rangle = \frac{\Omega^4 k^2\, e^2}{2560\pi^2 r_0^2}\Big[2a + (b - 3a)\cos^2(\Omega t)\Big]\,. \tag{A.4}$$

An estimate for the energy production rate on the brane is then

$$\frac{P}{V_3} = \frac{d\rho}{dt} = \frac{\Omega^5 k^2 e^2}{1280\pi^2 r_0^2}(-b+3a)\cos(\Omega t)\sin(\Omega t)\,. \tag{A.5}$$

Notice that this scales like $e^2\,\Omega^5(2k/r)^2$, just as does our estimate in the main text for the energy deposition rate for the brane.

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